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APPLICATION OF THE PATANKAR–SPALDING FINITE DIFFERENCE PROCEDURE TO TURBULENT RADIATING BOUNDARY LAYER FLOW

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Abstract—The Patankar/Spalding finite difference solution procedure has been modified to account for the transport of radiant energy in a turbulent optically thin boundary layer. The resulting procedure has been applied to high speed external flows. Results show that radiation losses at a Mach number of 25 and sea-level conditions do not significantly alter temperature profiles or heating rates while at Mach 40, radiation cooling reduces heating rates by approximately 20 per cent.

NOMENCLATURE

- B, radiosity;
- Cp, specific heat, constant pressure;
- H, stagnation enthalpy;
- Pr, Prandtl number;
- q, heat flux;
- T, temperature (Rankine);
- u, velocity, x-direction;
- v, velocity, y-direction;
- x, length, parallel to body surface;
- y, length, normal to body surface;
- κ_p, Planck mean absorption coefficient;
- μ_{eff}, effective viscosity;
- ρ, density;
- ρ_0, standard density;
- σ, Stefan–Boltzmann radiation constant;
- ϕ, dependent variable in finite difference equations;
- Φ, generation term for ϕ;
- ψ, stream function;
- ω, dimensionless cross-stream variable.

Subscripts

- C, convective term;
- d, viscous dissipation term;
- D, downstream point in the finite difference grid;
- R, radiation term;
- U, upstream point in the finite difference grid;
- W, body surface condition;
- ∞, free-stream condition;
- λ, wavelength.

INTRODUCTION

THE FINITE difference solution technique reported by Patankar and Spalding [1] is applied successfully to the problem of a turbulent radiating boundary layer. The Patankar–Spalding technique is adapted to this type of flow by specifying the radiative flux term in the energy equation as part of the *source term*, identified as Φ, in [1].

This note considers the flow in air past a flat plate at zero angle of attack. The boundary layer is assumed to be optically thin and in thermodynamic equilibrium; the air is gray with respect to thermal radiation.

ANALYSIS

For the sake of brevity, consider only the governing equations for the boundary layer on the flat plate.

Continuity

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{1}$$

Momentum

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\mu_{\text{eff}} \frac{\partial u}{\partial y}\right) \tag{2}$$

Energy

$$\rho u \frac{\partial H}{\partial x} + \rho v \frac{\partial H}{\partial y} = \frac{\partial}{\partial y} \times \left[\frac{\mu_{\text{eff}}}{Pr}\frac{\partial H}{\partial y} + \mu_{\text{eff}}\left(1 - \frac{1}{Pr}\right)\frac{1}{2}\frac{\partial u^2}{\partial y}\right] - \operatorname{div} \bar{q}_R. \tag{3}$$

With the introduction of the two-dimensional stream function ψ, defined as

$$\frac{\partial \psi}{\partial y} = \rho u \tag{4}$$

$$\frac{\partial \psi}{\partial x} = -\rho v \tag{5}$$

the continuity equation (1) is satisfied. Following the Patankar–Spalding development [1], a dimensionless cross-stream variable, ω, is defined as

$$\omega \equiv \frac{\psi - \psi_W}{\psi_\infty - \psi_W} \tag{6}$$

and equations (2) and (3) can then be transformed into the common form

$$\frac{\partial \phi}{\partial x} + (a + b\omega)\frac{\partial \phi}{\partial \omega} = \frac{\partial}{\partial \omega}\left(c\frac{\partial \phi}{\partial \omega}\right) + \frac{\Phi}{\rho u} \tag{7}$$

where $\phi = u$ or H and

$$\Phi = \frac{\partial}{\partial y}\left[\mu_{\text{eff}}\left(1 - \frac{1}{Pr}\right)\frac{1}{2}\frac{\partial u^2}{\partial y}\right] - \operatorname{div} \bar{q}_R = \Phi_d + \Phi_R \tag{8}$$

in the energy equation. In equation (8),

$$\Phi_d = \frac{\partial}{\partial y}\left[\mu_{\text{eff}}\left(1 - \frac{1}{Pr}\right)\frac{1}{2}\frac{\partial u^2}{\partial y}\right] \tag{9}$$

and

$$\Phi_R = -\operatorname{div} \bar{q}_R. \tag{10}$$

The inclusion of the radiation generation (or source) term, Φ_R, is the only alteration of the compressible boundary layer equations presented in [1]. Since the viscous dissipation term is successfully treated by Patankar and Spalding, the discussion here is confined to Φ_R, which is not included in their work.

In developing the expression for the radiation source term, $-\operatorname{div} \bar{q}_R$, the following assumptions are made:

1. A one-dimensional radiation regime exists;
2. The boundary layer is optically thin;
3. The boundary layer is gray with respect to thermal radiation;
4. The energy emitted at the wall and by the freestream is emitted as black body radiation at the respective temperatures of each location.

The governing equation for an optically thin, one-dimensional medium (with no further restrictions) is given by Sparrow and Cess [2]:

$$-\frac{d}{dy} q_{R_\lambda} = 2\kappa_\lambda [B_{1_\lambda} + B_{2_\lambda} - 2e_{b_\lambda}(y)]. \tag{11}$$

This equation is developed in [2] for a one-dimensional medium with bounding surfaces 1 and 2. However, as is pointed out by Sparrow and Cess, the boundary layer is only a portion of the entire flow field; an adjacent region exists that is not optically thin but within which the temperature gradients are small. In this analysis, the thermal interaction between the boundary layer and the freestream is neglected and the boundary layer edge is assumed a bounding surface at a temperature, T_∞.

Since the boundary layer has also been assumed to be gray, the absorption coefficient is of course then independent of wavelength. Equation (11) then becomes, writing $\kappa_\lambda = \kappa$

$$-\frac{d}{dy} q_{R_\lambda} = 2\kappa [B_{1_\lambda} + B_{2_\lambda} - 2e_{b_\lambda}(y)]. \tag{12}$$

Integrating over all wavelengths, equation (12) becomes

$$-\operatorname{div} \bar{q}_R = -\int_0^\infty \frac{\mathrm{d}}{\mathrm{d}y} q_{R_\lambda} \,\mathrm{d}\lambda = 2\kappa \int_0^\infty \times [B_{1_\lambda} + B_{2_\lambda} - 2e_{b_\lambda}(y)] \,\mathrm{d}\lambda. \quad (13)$$

Since the wall and freestream are assumed to be black, bounding surfaces at their respective temperatures,

$$B_{1_\lambda} = e_{b1_\lambda} = e_{bw\lambda} \quad \text{and} \quad B_{2_\lambda} = e_{b_{2\lambda}} = e_{b\infty_\lambda},$$

using standard boundary layer nomenclature. Carrying out the integration of equation (13).

$$-\operatorname{div} \bar{q}_R = 2\kappa \int_0^\infty [e_{bw\lambda} + e_{b\infty\lambda} - 2e_{b_\lambda}(y)]\mathrm{d}\lambda = 2\kappa[e_{b_W} + e_{b_\infty} - 2e_b(y)] = 2\kappa [\sigma T_W^4 + \sigma T_\infty^4 - 2\sigma T^4(y)]. \quad (14)$$

The Planck mean absorption is defined [2] as,

$$\kappa_p(T) = \frac{\int_0^\infty \kappa_\lambda e_{b_\lambda} \,\mathrm{d}\lambda}{e_b} \quad (15)$$

but under the gray assumption, κ is independent of wavelength; hence,

$$\kappa_p(T) = \frac{\kappa \int_0^\infty e_{b_\lambda} \mathrm{d}\lambda}{e_b} = \kappa \frac{e_b}{e_b} = \kappa(T) \quad (16)$$

and equation (14) may now be written

$$-\operatorname{div} \bar{q}_R = 2\sigma\kappa_p(T)[T_W^4 + T_\infty^4 - 2T^4(y)]. \quad (17)$$

Equation (17) may be understood physically in that the first two terms represent local absorption in an elemental volume owing to energy that originated at the bounding surfaces; the last term represents local emission by the element. To employ equation (17) in the Patankar–Spalding technique, the quantity

$$\frac{\Phi_R}{\rho u} = \left.\frac{\Phi_R}{\rho u}\right|_U + \left.\frac{\partial}{\partial \phi}\left(\frac{\Phi_R}{\rho u}\right)\right|_U (\phi_D - \phi_U) \quad (18)$$

must be evaluated. For the energy equation, it follows that

$$\left.\frac{\partial}{\partial \phi}\left(\frac{\Phi_R}{\rho u}\right)\right| = \frac{\partial}{\partial H} \times \left.\left[\frac{2\sigma\kappa_p(T)}{\rho u}(T_W^4 + T_\infty^4 - 2T^4)\right]\right|_U \quad (19)$$

or

$$\left.\frac{\partial}{\partial \phi}\left(\frac{\Phi_R}{\rho u}\right)\right|_U = \frac{2\sigma}{C\rho u}\left\{(T_W^4 + T_\infty^4 - 2T^4) \times \frac{\partial}{\partial T}\left[\frac{\kappa_p(T)}{\rho(T)}\right] - \frac{\kappa_p(T)}{\rho(T)}[8T^3]\right\}\Bigg|_U. \quad (20)$$

Assuming that the density of air varies inversely with temperature and using the absorption coefficient suggested by Traugott [3], that is

$$\kappa_p(T) = 115{\cdot}0 \frac{\rho}{\rho_0}\left(\frac{T}{10^4}\right)^5 FT^{-1} \quad (21)$$

then

$$\frac{\partial}{\partial T}\left[\frac{\kappa_p(T)}{\rho(T)}\right] = \frac{0{\cdot}0575}{\rho_0}\left(\frac{T}{10^4}\right)^4 \quad (22)$$

and

$$\left.\frac{\partial}{\partial \phi}\left(\frac{\Phi_R}{\rho u}\right)\right|_U = \frac{2\sigma}{C\rho u}\left[\frac{0{\cdot}0575}{\rho_0}\left(\frac{T}{10^4}\right)^4(T_W^4 + T_\infty^4 - 2T^4) - 8\frac{\kappa_p(T)}{\rho(T)}T^3\right]\Bigg|_U. \quad (23)$$

Therefore, in equation (18), all values evaluated at U, in the finite difference procedure, are known either from the initial profile specification or from the previous iteration. Thus, ϕ_D is evaluated as the unknown.

To solve the radiating problem, equations (18) and (23) are included in the general difference equation of the Patankar–Spalding method. Hypersonic flows at Mach 25 and 40, with and without thermal radiation effects, are analyzed. The initial velocity and enthalpy profiles are specified as linear with respect to cross-stream distance, although after a very few iterative steps, the solutions are nearly insensitive to the form of the initial profiles.

Another departure from the Patankar–Spalding procedure involves the calculation of the heat transferred from the fluid to the wall by conduction. Instead of using wall-flux relationships developed by Patankar and Spalding, the actual temperature gradient is calculated at the wall, and the heat flux is thus determined.

The radiative heat flux at the wall is calculated by integrating equation (17) as follows

$$q_{R\,\mathrm{Total}} = \int_0^{\delta} 2\sigma\kappa_p(T)[T_W^{\,4} + T_\infty^{\,4} - 2T^4(y)]\mathrm{d}y \qquad (24)$$

to give the total radiant energy emitted from the boundary layer. Assuming the plate surface to be black, the radiative heat transfer at the wall is given by

$$q_{R\,\mathrm{Wall}} = \tfrac{1}{2} q_{R\,\mathrm{Total}} - \sigma T_W^{\,4}. \qquad (25)$$

RESULTS

Preliminary results are compared on a qualitative basis with solutions in the literature. Figure 1 contains dimensionless temperature profiles for a flat plate in air at $M_\infty = 25$. Clearly, the presence of *optically thin* radiation does not appreciably alter the dimensionless temperature profiles.

Figure 2 shows that the presence of radiation does in fact alter the temperature distribution

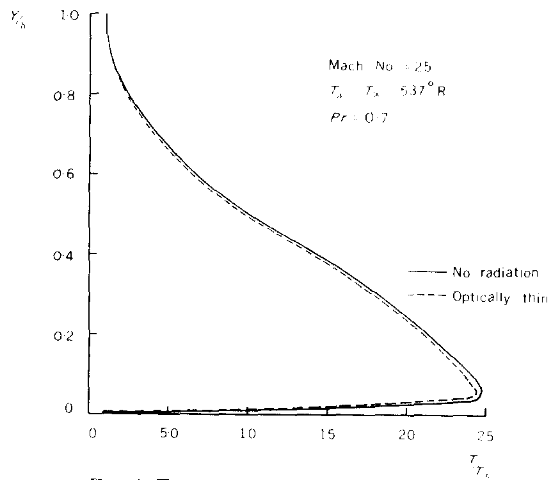

FIG. 1. Temperature profiles, $X = 0{\cdot}023$ ft.

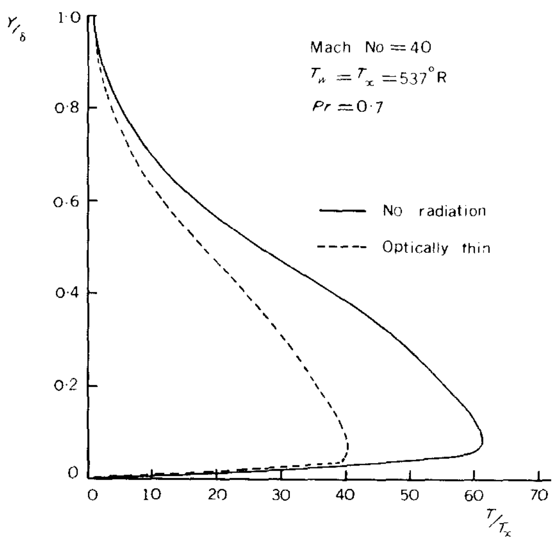

FIG. 2. Temperature profiles, $X = 0{\cdot}023$ ft.

Table 1. Comparison of convective heating rates for radiating and nonradiating cases; $X = 0{\cdot}023\,ft$

M_∞	$q^*_{C\,\mathrm{Wall}}$	$q_{R\,\mathrm{Wall}}$	$q_{\mathrm{Total\,wall}}$	Per cent decrease resulting from radiative cooling (convection)	Per cent decrease resulting from radiative cooling (total)	Conditions
25	$3{\cdot}412 \times 10^3$	0·0	$3{\cdot}412 \times 10^3$			No radiation
	$3{\cdot}343 \times 10^3$	4·17	$3{\cdot}347 \times 10^3$	2·02	1·90	Optically thin
40	$8{\cdot}259 \times 10^3$	0·0	$8{\cdot}259 \times 10^3$			No radiation
	$6{\cdot}205 \times 10^3$	250·5	$6{\cdot}455 \times 10^3$	24·87	21·84	Optically thin

* Heat flux units: Btu/ft^2-s.

in the boundary layer for a free-stream Mach number of 40. Radiation cooling decreases the convective heat transfer to the wall by 24·87 per cent and the total heat transfer by 21·84 per cent from the nonradiating case, as shown in Table 1.

These results tend to conflict with the laminar analysis of Koh and De Silva [4], who found that radiation had no effect on the convection at the wall for $M_\infty = 40$. This discrepancy is easily resolved when the differences in turbulent and laminar viscous forces are considered. The shearing forces in turbulent layers are approximately proportional to the square of the mean velocity while in laminar flow, a first power relationship exists [5]. Therefore, viscous heating in the turbulent case is much greater than in the laminar cases. Consequently, there is a greater energy loss caused by radiation emitted at a higher temperature from air within the turbulent boundary layer than in the laminar layer for the same free-stream conditions. The effect on the net heat transfer, that is, radiation and convection, is seen to be comparable to convection alone since the total radiative heat flux at the wall is small compared to the convection.

CONCLUSION

The purpose of this communication is to extend the usefulness of a state-of-the-art contribution—the Patankar–Spalding finite difference technique—to a different problem area. The ease with which this method can be extended to optically thin radiating flows is clearly demonstrated. Its obvious versatility and rapid solution times* have prompted the authors to attempt to incorporate the full integral radiation terms in the energy equation, including the non-gray and temperature dependent absorption coefficients. The inherent speed of this numerical solution may eliminate the necessity for long computer runs when integral terms are included, thus eliminating a major drawback to exact numerical solutions of the radiating boundary layer.

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* Computer times, with the radiation terms included, using only Source Modules, average approximately 5 min on the IBM System 360.

APPLICATION DE LA PROCÉDURE AUX DIFFÉRENCES FINIES DE PATANKAR-SPALDING À L'ÉCOULEMENT TURBULENT À COUCHE LIMITE RAYONNANTE

Résumé—La procédure aux différences finies de Patankar–Spalding a été modifiée pour tenir compte du transport d'énergie rayonnante dans une couche limite turbulente optiquement mince. La procédure résultante a été appliquée aux écoulements externes à grande vitesse. Les résultats montrent que les pertes par rayonnement à un nombre de Mach égal à 25 dans les conditions standart au niveau de la mer ne modifient pas de façon significative les profils de température et les flux thermiques tandis qu'à un nombre de Mach égal à 40, le refroidissement par rayonnement réduit les flux thermiques d'à peu près 20 pour cent.

ANWENDUNG DES DIFFERENZENVERFAHRENS NACH PATANKAR/SPALDING AUF EINE TURBULENTE GRENZSCHICHTSTRÖMUNG MIT STRAHLUNGSEINFLUSS

Zusammenfassung—Das Differenzenverfahren nach Patankar–Spalding wurde so modifiziert, dass der Fall des Energietransports durch Strahlung in einer turbulenten, optisch dünnen Grenzschicht mit berücksichtigt wird. Das resultierende Verfahren wurde auf Aussenströmungen bei hohen Geschwindigkeiten angewandt. Die Ergebnisse zeigen, dass die Strahlungsverluste bei einer Mach-Zahl von 25 und Meereshöhenbedingungen die Temperaturprofile oder die Wärmeströme nicht wesentlich verändern, während bei einer Mach-Zahl von 40 letztere durch Strahlungskühlung um ungefähr 20 Prozent reduziert werden.

ПРИМЕНЕНИЕ МЕТОДА КОНЕЧНЫХ РАЗНОСТЕЙ ПАТАНКАРА-СПОЛДИНГА К ТУРБУЛЕН-тному ПОГРАНИЧНОМУ СЛОЮ С ИЗЛУЧЕНИЕМ

Аннотация—Метод конечных разностей Патанкара-Сполдинга модифицирован для учета переноса лучистой энергии в турбулентном оптиуески тонком пограничном слое. Этот модифицированный метод применяется для расчета высокоскоростных внещних потоков. Результаты показывают, что потери за счет излучения при числе Маха, равном 25, и давлениях, соответствующих давлению на уровне мора, не изменяют значительно температурные профили или скорости нагрева, тогда как при М = 40 лучистое охлаждение уменьшает скорости нагрева приблизительно на 20%.